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Lipschitzian Selections in Approximation from Nonconvex Sets of Bounded Functions*

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Consider the problem of finding a best uniform approximation to a function f from a nonconvex set K in the space of bounded functions. Conditions are developed on K so that the operator mapping f to one of its best approximations f' is Lipschitzian with some constant C and is optimal Lipschitzian, i.e., has the smallest C among all such operators.

1. INTRODUCTION

Consider the space of bounded functions on a set S with uniform norm $\|\cdot\|$. A best approximation to a function f from a nonconvex set K in this space is not unique in general. An operator $T: f \to f'$, where f' is a best approximation to f, is a Lipschitzian selection oprator if $\|f'-h'\| \leqslant C(T)\,\|f-h\|$ for all f, h and some least number $C(T)$. Such an operator is optimal if $C(T) \leqslant C(T')$ for all such Lipschitzian T' mapping each f to one of its best approximations. In this article, conditions are developed in some generality on the approximating set K so that Lipschitzian operators, optimal or otherwise, can be identified and their uniqueness determined. Concepts of epigraphic sets and maps are introduced and used in analysis. The problem considered is conceptually similar to the well-known problem of finding continuous selections.

Let S be any set and B denote the Banach space of bounded functions f on S with uniform norm $\|f\| = \sup\{|f(s)| : s \in S\}$. Let $K \subset B$ be a nonempty and nonconvex (i.e., not necessarily convex) set. Given an f in B, let $\Delta(f)$ denote the infimum of $\|f-k\|$ for k in K. The problem is to find an f' in K, called a best approximation to f from K, so that

$$\Delta(f) = \|f-f'\| = \inf\{\|f-k\| : k \in K\}. \tag{1.1}$$

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In general, A_f, the set of all best approximations to f, is not singleton. A Lipschitzian selection operator (LSO) T is a nonlinear operator which maps each f in B to an f' in A_f and satisfies, for some least number $C(T)$,

$$\|T(f) - T(h)\| \leqslant C(T) \, \|f - h\|,$$

for all f, h in B. T is an optimal Lipschitzian selection operator (OLSO) if $C(T) \leqslant C(T')$ for all LSOs T'. In this article we obtain conditions on K so that LSOs and OLSOs can be identified.

We now state three conditions on K; not all conditions will be imposed in every case under consideration.

(i) If $k \in K$ then $k + c \in K$ for all real c.

(ii) If $K' \subset K$ is a set of functions uniformly bounded above on S, then the function k', which is the pointwise supremum of functions in K', is in K.

(iii) If $K' \subset K$ is a set of functions uniformly bounded below on S, then the function k', which is the pointwise infimum of functions in K', is in K.

We now summarize our results and method of analysis. In Section 3, under conditions (i) and (ii) on K, we identify an LSO T with $C(T) = 2$. A symmetric result holds when conditions (i) and (iii) hold for K. If K satisfies all three conditions and K is convex, then an LSO T_λ is identified with $C(T_\lambda) = 1 + |2\lambda - 1|$ for each $0 \leqslant \lambda \leqslant 1$. In particular, when $\lambda = \frac{1}{2}$, the corresponding $T = T_\lambda$ is an OLSO with $C(T) = 1$. Another problem is also analyzed in Section 3. Given f in B, let $K_f = \{k \in K: k \leqslant f\}$ and $\bar{\Delta}(f)$ denote the infimum of $\|f - k\|$ for k in K_f. The problem is to find an f' in K_f so that

$$\bar{\Delta}(f) = \|f - f'\| = \inf\{\|f - k\| : k \in K_f\}. \tag{1.2}$$

Under conditions (i) and (ii) on K for the above problem, we identify an OLSO T with $C(T) = 1$ and show that it is unique. A symmetric result holds for a problem symmetric to (1.2) when K satisfies (i) and (iii). For the purpose of analysis, we introduce epigraphic sets and maps in Section 2. A set $U \subset S \times R$ is called epigraphic if the projection of U on S is S and the function f on S defined by $f(s) = \inf\{x: (s, x) \in U\}$, where $s \in S$, is in B. An epigraphic map has epigraphic sets for its domain and range. We define an epigraphic map A and a Hausdorff metric like function d on the subsets of $S \times R$ so that A is nonexpansive with respect to d. These mappings play a key role in analysis.

Conditions (i) and (ii) hold, for example, for convex and quasi-convex functions in B. The latter functions are those that satisfy $k(\lambda s + (1 - \lambda) t) \leqslant$

$\max\{k(s), k(t)\}$ for all s, t in a convex set $S \subset R^n$, all $0 \leqslant \lambda \leqslant 1$ [5]. All three conditions hold, for example, for monotone nondecreasing and, more generally, isotone functions on a partially ordered set. The problem of finding continuous selections has generated much interest in the literature. For surveys see [1, 2, 9]. However, not much is known about Lipschitzian selections; some references appear in the above mentioned surveys. OLSOs are identified in [7, 8] for the problem of approximation by quasi-convex and convex functions, and generalized isotone optimization. The results of this article are extended to the space of continuous functions in [9].

2. Epigraphic Sets and Maps

In this section we derive some key results concerning epigraphic maps.

For f in B define $K_f = \{k \in K: k \leqslant f\}$ and $K'_f = \{k \in K: k \geqslant f\}$. We observe that if K satisfies condition (i), then K_f and K'_f are not empty. Indeed, if $g \in K$ and $\delta = \|g - f\|$, then $g + \delta \geqslant f \geqslant g - \delta$. Hence $g - \delta \in K_f$ and $g + \delta \in K'_f$. Now, for f in B, let

$$\bar{f}(s) = \sup\{k(s): k \in K_f\}, \qquad s \in S,$$

and

$$\underline{f}(s) = \inf\{k(s): k \in K'_f\}, \qquad s \in S.$$

Note that $\bar{f}$ and $\underline{f}$ are in K if K satisfies, respectively, conditions (ii) and (iii). Clearly, $\bar{f} \leqslant f \leqslant \underline{f}$. The functions $\bar{f}$ and $\underline{f}$ are called, respectively, the greatest K-minorant and the smallest K-majorant of f. Note that $-\underline{f}$ is the greatest $-K$-minorant of $-f$. These functions are used for identifying best approximations in Section 3 and defining epigraphic maps below.

We denote the elements of $S \times R$ by (s, x), (t, y) where $s, t \in S$ and $x, y \in R$. For any f in B, let $E(f)$ denote the epigraph of f [5, 6], viz.,

$$E(f) = \{(s, x) \in S \times R: x \geqslant f(s)\}.$$

Motivated by this definition, we call $U \subset S \times R$ epigraphic if $\{s: (s, x) \in U\} = S$ and the function f defined by $f(s) = \inf\{x: (s, x) \in U\}$, $s \in S$, is in B. In this case, we say that U generates f. We are only concerned with the behavior of U at its "lower boundary." Note that $E(f)$ is epigraphic and generates f. An epigraphic map is defined to be a map whose domain and range are epigraphic subsets of $S \times R$. Assuming K satisfies conditions (i) and (ii), which ensures existence of $\bar{f}$ for any f in B, we now define an epigraphic map A as follows: If U is epigraphic and generates f, then $A(U) = E(\bar{f})$. To investigate properties of A, we define a function d.

Let $u=(s, x)$ and $v=(t, y)$ be elements of $S \times R$, and let

$$\begin{aligned} d'(u, v) &= |x-y|, && \text{if } s=t, \\ &= \infty, && \text{otherwise.} \end{aligned}$$

Let $U \subset S \times R$, $r \geqslant 0$, and define

$$B_r(U)=\{u \in S \times R: \inf\{d'(u, v): v \in U\} \leqslant r\}.$$

Analogous to the Hausdorff metric [3], define

$$d(U, V)=\inf\{r: U \subset B_r(V) \text{ and } V \subset B_r(U)\},$$

where $U, V \subset S \times R$. Clearly, $0 \leqslant d \leqslant \infty$. The function d was also used in [8]. It is easy to see that $f, h \in B$, then

$$d(E(f), E(h))=\|f-h\| \tag{2.1}$$

and

$$d(A(E(f)), A(E(h)))=\|\bar{f}-\bar{h}\|. \tag{2.2}$$

Note that if U is epigraphic and generates f, then $B_r(U)$ is epigraphic and generates $f-r$. The following lemma gives properties of A.

Lemma 2.1. *Let U and V be epigraphic sets.*

(a) *If $U \subset V$ then $A(U) \subset A(V)$.*

(b) $A(B_r(U))=B_r(A(U))$.

Proof. Let U and V generate f and h, respectively. Then $f \geqslant h$. Hence, $\bar{f} \geqslant \bar{h}$ and (a) follows. To establish (b), we note that both sides of (b) equal $E(\bar{f}-r)$. The proof is complete.

Proposition 2.1. *A is nonexpansive with respect to d, i.e.,*

$$d(A(U), A(V)) \leqslant d(U, V)$$

holds for all epigraphic sets U and V.

Proof. If $r \geqslant 0$, $U \subset B_r(V)$, and $V \subset B_r(U)$, then by Lemma 2.1 we have

$$A(U) \subset A(B_r(V))=B_r(A(V)).$$

Similarly, $A(V) \subset B_r(A(U))$. From the definition of d, the required conclusion follows. The proof is complete.

The following is an application of the above proposition.

PROPOSITION 2.2. *If K satisfies conditions* (i) *and* (ii) *then* $\|\bar{f}-\bar{h}\| \leqslant \|f-h\|$ *for all f, h in B. Similarly, if K satisfies* (i) *and* (iii), *then* $\|\underline{f}-\underline{h}\| \leqslant \|f-h\|$.

Proof. If K satisfies (i) and (ii), then $\bar{f}$ exists for each f and the epigraphic map A is well defined. Proposition 2.1 with $U=E(f)$, $V=E(h)$, and (2.1), (2.2) establish the first inequality of the proposition. If K satisfies (i) and (iii), then $-K$ satisfies (i) and (ii). Again, $-\underline{f}$ is the greatest $-K$-minorant of $-f$. Thus, the second inequality follows from the first by substituting $-f$ and $-\underline{f}$ for f and $\bar{f}$, respectively. The proof is complete.

3. LIPSCHITZIAN SELECTIONS

In this section we present our main results. An f' in K is the maximal (minimal) best approximation to f if $f' \geqslant g$ ($f' \leqslant g$) for all best approximations g to f. We state two theorems.

THEOREM 3.1. *The following applies to Problem* (1.1).

(a) *K nonconvex. If K satisfies conditions* (i) *and* (ii), *then*

$$\Delta(f)=\tfrac{1}{2}\|f-\bar{f}\| \tag{3.1}$$

and $f'=\bar{f}+\Delta(f)$ is the maximal best approximation to f. Furthermore if f, $h \in B$, then

$$\|f'-h'\| \leqslant \|f-h\|, \qquad \text{if } \Delta(f)=\Delta(h) \tag{3.2}$$

and

$$\|f'-h'\| \leqslant 2\,\|f-h\|. \tag{3.3}$$

The operator $T\colon B \to K$ defined by $T(f)=f'$ is a Lipschitzian selection operator with $C(T)=2$.

(b) *K noncovex. If K satisfies conditions* (i) *and* (iii), *then* (a) *holds with $\bar{f}$ replaced by $\underline{f}$ and $f'=\underline{f}-\Delta(f)$, which is the minimal best approximation to f.*

(c) *K convex. If K satisfies conditions* (i), (ii), *and* (iii), *then*

$$\Delta(f)=\tfrac{1}{2}\|f-\bar{f}\|=\tfrac{1}{2}\|f-\underline{f}\| \tag{3.4}$$

and $\bar{f}+\Delta(f)(\underline{f}-\Delta(f))$ *is the maximal (minimal) best approximation to* f. *Furthermore, a* g *in* K *is a best approximation to* f *if and only if* $\underline{f}-\Delta(f)\leqslant g\leqslant\bar{f}+\Delta(f)$. *Hence,*

$$f'=\lambda\bar{f}+(1-\lambda)\underline{f}+(2\lambda-1)\,\Delta(f),\qquad 0\leqslant\lambda\leqslant 1,$$

is a best approximation to f. *For all* f, $h\in B$, (3.2) *holds for this* f' *and*

$$\|f'-h'\|\leqslant(1+|2\lambda-1|)\,\|f-h\|,\qquad 0\leqslant\lambda\leqslant 1. \tag{3.5}$$

The operator $T_\lambda\colon B\to K$ *defined by* $T_\lambda(f)=f'$ *is a Lipschitzian selection operator with* $C(T_\lambda)=1+|2\lambda-1|$. *When* $\lambda=\frac{1}{2}$, *the operator* $T=T_\lambda$ *defined by* $T(f)=f'=\frac{1}{2}(\bar{f}+\underline{f})$ *is an optimal Lipschitzian selection operator with* $C(T)=1$.

THEOREM 3.2. *The following applies to Problem* (1.2) *for a nonconvex* K. *If* K *satisfies conditions* (i) *and* (ii), *then* $\bar{f}$ *is the maximal best approximation to* f *and* $\bar{\Delta}(f)=\|f-\bar{f}\|=2\Delta(f)$. *The operator* $T\colon B\to K$ *defined by* $T(f)=\bar{f}$ *is the unique optimal Lipschitzian selection operator with* $C(T)=1$.

It is easy to verify that for all Lipschitzian operators obtained in the above two theorems we have $T(f+c)=T(f)+c$ for all real c. Furthermore, if K is a cone (i.e., $\lambda f\in K$ whenever $f\in K$, $\lambda\geqslant 0$), then $T(\lambda f)=\lambda T(f)$, $\lambda\geqslant 0$. Now, we state a proposition which is used in the proof of the above theorems.

PROPOSITION 3.1. *The following applies to Problem* (1.1) *for a nonconvex* K.

(a) *If* K *satisfies conditions* (i) *and* (ii), *then* (3.1) *holds and* $f'=\bar{f}+\Delta(f)$ *is the maximal best approximation to* f.

(b) *If* K *satisfies conditions* (i) *and* (iii), *then* (3.1) *holds with* $\bar{f}$ *replaced by* $\underline{f}$ *and* $f'=\underline{f}-\Delta(f)$ *is the minimal best approximation to* f.

Proof. Proof of part (a) is identical to that of [8, Proposition 2.1], however, we give it for the convenience of the reader. Let $g\in K$ and $g_0=g-\|f-g\|$. Then $g_0\in K$ by condition (i) and $f\geqslant g_0$. Consequently, $f\geqslant\bar{f}\geqslant g_0$, which gives $f-\bar{f}\leqslant f-g+\|f-g\|$ or $\|f-\bar{f}\|/2\leqslant\|f-g\|$ for all g in K. Thus $\|f-\bar{f}\|/2\leqslant\Delta(f)$. If $f'=\bar{f}+\|f-\bar{f}\|/2$ then $f'\in K$ by condition (i). It is easy to verify that $\|f-f'\|\leqslant\|f-\bar{f}\|/2$. Hence (3.1) follows and f' is a best approximation to f. If g is any best approximation, then $f\geqslant g-\Delta(f)$, which is in K. Hence, $f\geqslant\bar{f}\geqslant g-\Delta(f)$ which gives $f'\geqslant g$. Thus, f' is the maximal best approximation. Part (b) may be established by symmetric arguments. The proof is complete.

Before proceeding to the proofs of the theorems, we observe that [4, p. 17] if $f, h \in B$, then

$$|\Delta(f) - \Delta(h)| \leqslant \|f - h\|. \tag{3.6}$$

Proof of Theorem 3.1. (a) By Proposition 3.1, $f' = \bar{f} + \Delta(f)$ is the maximal best approximation to f. Hence,

$$\|f' - h'\| \leqslant \|\bar{f} - \bar{h}\| + |\Delta(f) - \Delta(h)|,$$

which, together with Proposition 2.2, establishes (3.2). Again, the above inequality, (3.6), and Proposition 2.2 establish (3.3). By (3.3), $C(T) \leqslant 2$. To show $C(T) = 2$, let $S = [0, 1]$ and let K be the set of all convex functions on S. Let $f(0) = -1$, $f(s) = 1$ on $(0, 1]$, and $h = 0$ on S. Then $f'(s) = 2s$, $h' = 0$, $\|f' - h'\| = 2$, and $\|f - h\| = 1$. Hence $C(T) = 2$.

(b) Proof of this part is similar to that of (a).

(c) The assertions concerning $\bar{f} + \Delta(f)$, $\underline{f} - \Delta(f)$ and the validity of (3.4) follow from (a) and (b). Since K is convex, $\lambda(\bar{f} + \Delta(f)) + (1 - \lambda)(\underline{f} - \Delta(f))$, which equals f', is a best approximation. Now

$$\|f' - h'\| \leqslant \lambda \, \|\bar{f} - \bar{h}\| + (1 - \lambda) \, \|\underline{f} - \underline{h}\| + |2\lambda - 1| \, |\Delta(f) - \Delta(h)|.$$

This inequality, (3.6), and Proposition 2.2 establish (3.2) and (3.5). Now $C(T_\lambda) \leqslant 1 + |2\lambda - 1|$. Let $S = [0, 1]$ and let K be the set of all nondecreasing functions on S. Let $f(s) = -1$ for $s = 0, \frac{1}{2}$, and $f(s) = 1$, otherwise. Also let $h = 0$ on S. Using functions f and h, one may easily show that equality holds in (3.5) for all $0 \leqslant \lambda \leqslant 1$. Hence $C(T_\lambda) = 1 + |2\lambda - 1|$. This example appears in [7, p. 217]. It remains to show that T is an OLSO. Clearly $C(T) \leqslant 1$. Let $f \in K$ and $h = f - c$ where $c > 0$. Then $h \in K$. If T' is any LSO, then $T'(f) = f$ and $T'(h) = h$. Since

$$\|T'(f) - T'(h)\| \leqslant C(T') \, \|f - h\|,$$

we have $C(T') \geqslant 1$. Hence $C(T) = 1$ is the minimum value of $C(T')$ for all T'. Thus T is an OLSO. The proof is now complete.

Proof of Theorem 3.2. The assertions concerning $\bar{f}$ and $\bar{\Delta}(f)$ follow immediately from the definition of $\bar{f}$ and (3.1). If $f, h \in B$, then by Proposition 2.2, we have $\|\bar{f} - \bar{h}\| \leqslant \|f - h\|$. Consequently $C(T) \leqslant 1$. To show T is an OLSO, let $f \in K$ and $h = f - c$ where $c > 0$. If T' is any LSO then $T'(f) = f$ and $T'(h) = h$. Then, as in the proof of Theorem 3.1(c), we have $C(T') \geqslant 1$. Hence T is an OLSO. To show uniqueness of T, let T' be any OLSO and $f \in B$. We show that $T'(f) = \bar{f}$. Let $T'(f) = f'$. Since $\bar{f}$ is the maximal best approximation to f, we have $f' \leqslant \bar{f}$. Let $h = \bar{f} + c$ where $c = \|f - \bar{f}\|$. Then $0 \leqslant h - f \leqslant h - \bar{f} = c$. Also $h \in K$ and hence $T'(h) = h$.

Since T' is an OLSO we have $\|T'(h)-T'(f)\| \leqslant \|h-f\|$ which gives $\|\bar{f}+c-f'\| \leqslant c$. It follows from $f' \leqslant \bar{f}$ that $f'=\bar{f}$. The proof is now complete.

References

1. F. Deutsch, A survey of metric selections, *Contemp. Math.* **18** (1983), 49–71.
2. F. Deutsch, An exposition of recent results on continuous metric selections, *in* "Numerical Methods of Approximation Theory," (L. Collatz, G. Meinardus, and G. Nürnberger, Eds.), ISNM, 81, Birkhaüser Verlag, Basel, 1987, pp. 67–79.
3. J. L. Kelley, "General Topology," Van Nostrand, Princeton, NJ, 1955.
4. N. P. Korneichuk, "Extremal Problems in Approximation Theory," Nauka, Moscow, 1976.
5. A. W. Roberts and D. E. Varberg, "Convex Functions," Academic Press, New York, 1973.
6. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
7. V. A. Ubhaya, Lipschitz condition in minimum norm problems on bounded functions, *J. Approx. Theory* **45** (1985), 201–218.
8. V. A. Ubhaya, Uniform approximation by quasi-convex and convex functions, *J. Approx. Theory* **55** (1988), 326–336.
9. V. A. Ubhaya, Lipschitzian selections in best approximation by continuous functions, *J. Approx. Theory*, in press.
10. D. Yost, Best approximation operators in functional analysis, *Proc. Centre Math. Anal. Austral. Nat. Univ.* **8** (1984), 249–270.